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Gravity in Hyperspin Manifolds

Shlomit Ritz Finkelstein[1]

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We seek the dynamics of a Bergmann manifold: a manifold of dimension $n = N^2$ supporting a bundle of spinor spaces of dimension N, and a map σ from the tangent spaces to the Hermitian spinor forms. Even though the spin-vector σ is the fundamental variable of the theory, every invariant analytic function depending on σ and its first m derivatives alone can be expressed in terms of the chronometric tensor g and its first m derivatives. Bergmann manifolds of dimension $n > 4$ do not have invariant second-order equations for σ. We find a family of invariant actions which lead to nth-order quasilinear equations of motion on Bergmann manifolds and reduce to the Einstein-Hilbert action for $n = 2$. The resulting gauge particles have spin, 1/2, 1, 3/2, and 2.

1. INTRODUCTION

This work is a segment of a large program to provide a unified theory of the forces of nature and of the internal degrees of freedom of particles, such as spin, color, and flavor. It also unites two different approaches to this goal. One approach is through higher dimensional theories (Kaluza-Klein theories) in which electromagnetism and other forces emerge as manifestation of the extra dimensions. Another is to assume at each time-space point an underlying spinor space that is "soldered" to the tangent space at this point. This soldering is taken as the fundamental variable of the theory and the time-space manifold arises from it.

Finkelstein (1986) marries the two approaches into one theory in which the manifold emerges from an underlying spinorial space, with spinors which are not two-dimensional as customary, but can be of any dimension $N > 0$. The manifold is locally diffeomorphic to the space of Hermitian forms on a spinor space, so that its dimension is $n = N^2$. It turns out that the manifold is not Riemannian. The norm of proper time (except for $N = 2$, $n = 4$) is not quadratic, but N-ic. The chronometric tensor g carries N

[1]Georgia Institute of Technology, Atlanta, Georgia 30332.

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indices rather than two. [A more detailed presentation of the above ideas is to be found in Finkelstein *et al.* (1986).] Since Bergmann (1957) was the first to suggest spinors (with $N=2$) as primary elements of time-space, the manifolds bear his name and B_N stands for a manifold based on an N-dimensional spinor space. We use the prefix *hyper* on these manifolds (hyperspin, hyperaction, hypergravity, etc.)

It is clear that the program of D. Finkelstein calls for a generalization of general relativity from 2- to N-component spinors. The present work is one step in this program. It studies the dynamics of the generalized theory.

The first goal was to find a suitable action or family of actions for Bergmann manifolds. Several actions on higher dimension Riemannian manifolds are suggested by Lovelock (1971) and Zumino (1986) independently. [See also a survey by Deruelle and Madore (1986) of papers concerning Lovelock's action.]

Section 2 describes attempts to generalize those higher dimension actions to hypergravity. In doing so, a deeper understanding of Bergmann manifolds is achieved. In Section 2.2 the concept of quasitensor is defined. The central theorem of Section 2.3 deals with the fact that the spin-vector σ does not show in the action of general relativity. It proves that even though the fundamental variable of the theory is the spin-vector σ, every invariant time-space function of σ and its first m derivatives is such a function of g and its first m derivatives alone. Section 2.4 studies invariant tensors on Bergmann manifolds and proves that they are a sum of permutations of products of the Kronecker delta with the Grassmann form, the fundamental antisymmetric N-index tensor of the spinor space, defined by a fundamental scalar density ρ. The search for an action in the spirit of Lovelock and Zumino comes to its end in Theorem 2.5.2, which proves that invariant second-order equations exist only for a 4-dimensional Bergmann manifold, and in Theorem 2.5.3, which proves that Zumino's actions do not generalize to second-order actions for Bergmann manifolds with dimension higher than four.

In Section 3 a family of actions on Bergmann manifolds is found and it is proved in Section 3.1 that for $N=2$ the hyperactions are proportional to the Einstein-Hilbert action. The equations of motion are calculated from the variational principle in Section 3.2. Section 3.3 gives the equation of motion on a Bergmann manifold with sources. The equations resemble Einstein field equations and reduce to them for $N=2$. The spin-vector is hidden and the explicit dependence of both the action and the equations of motion is on the chronometric tensor g and its derivatives alone. Then a flat background is assumed with a small perturbation $\delta\sigma$ to describe gauge particles. The spin of the particles is discussed in Section 3.4 and a spin spectrum given by D. Finkelstein is corrected. Section 3.5 states some questions to be answered in the future concerning uniqueness of the action, solutions of the equations of motion, collapse of the internal dimensions, and black and white holes. These are representatives of a large fund of still open questions related to gravity on Bergmann manifolds.

Section 3 concludes the presentation with a brief examination of the major points that have been accomplished.

2. SEARCH FOR ACTIONS IN BERGMANN MANIFOLDS

2.1. Introduction

In this section we answer two questions about Bergmann manifolds.

In principle, the field variable of an action functional for hypergravity is the hyperspin vector σ. First we ask if all action densities for hypergravity that are analytic invariant functions of σ and its first m derivatives may be expressed as functions of g and its first m derivatives alone. The answer is yes. Using only time-space (nonspinorial) variables to make the action does not involve loss of generality, even though the underlying structure is spinorial.

The case $N = 2$, $n = 4$, $m = 2$ is treated by Noriega and Schifini (1986). Our proof is for arbitrary spinor dimension N and arbitrary order of derivative m, and requires a different method.

We then discuss possible actions and ask if there is an analogue of the Einstein-Hilbert action, an invariant action of the second differential order that is linear in the second derivative of σ (or g). The answer is no, except in four dimensions. Among Bergmann manifolds of all dimensions, only the four-dimensional ones have an invariant action principle leading to a quasilinear second-order equation for the spin-vector (and therefore for the chronometric tensor). This is in contrast with the Riemannian theory, which admits increasingly many such actions with increasing dimension (Lovelock, 1971; Zumino, 1986).

2.2. Quasitensors

By *geometrical object* (Anderson, 1967) we mean an entity whose description, if known in one coordinate system, can be calculated in any other; briefly, a description with a transformation law, under a specified collection of transformations. The geometric objects that occur here are *quasitensors.* A quasitensor is defined as a geometric object that undergoes a linear transformation with coefficients possibly depending not only on the Jacobian matrix $X = (\partial x'/\partial x)$ of the coordinate transformation, but also on the first- and higher order coordinate derivatives of X.

Examples: Relative to (differentiable) coordinate transformations, the chronometric tensor g of a Riemannian manifold is a geometric object, while ∂g (where ∂ is an ordinary derivative) is not. The pair $(g, \partial g)$, however, is a geometric object, and a quasitensor. The same holds for Bergmann manifolds.

We consider entities that are scalars with respect to spinor transformations and quasitensors with respect to time-space transformations. We call them *time-space quasitensors.*

If a functional relation $Y=f(X)$ between geometric objects X and Y commutes with the transformation T

$$f(TX)=Tf(X)$$

then it is called invariant under T.

Definition. We designate the mth derivative schematically by

$$\partial^m=\partial_{a_1}\cdots\partial_{a_m}$$

and the set of all derivatives up to and including the Mth by

$$\partial^{\{M\}}\sigma=\subset\partial^m\sigma|m=0,1,2,\ldots,M\}$$

$\partial^{\{M\}}\sigma$ is a quasitensor for any $M=0,1,2,\ldots$; so is the collection $\partial^{\{M\}}g$.

2.3. Hiding the Hyperspin Vector

Theorem 2.3.1. At each point of an n-dimensional manifold, any analytic time-space quasitensor function of the quasitensor $\partial^{\{M\}}\sigma$ is a function of σ and the time-space quasitensor $\partial^{\{M\}}g$ alone.

Proof. For the case $N=2$, $m=2$ see Noriega and Schifini (1986).

It suffices to prove the above-claimed relationship between $\partial^m\sigma$ and $\partial^{\{m\}}g$. We do this by induction on the order m of the derivatives. We define the symmetric polyspinor

$$\delta^{\alpha_1\alpha_2\cdots\alpha_N}=\varepsilon^{A_1A_2\cdots A_N}\varepsilon^{\dot{A}_1\dot{A}_2\cdots\dot{A}_N}$$

Using a somewhat shorter index notation in which $\alpha=A\dot{A}$, we remember that

$$\mathrm{g}^{\mathrm{a}_1\mathrm{a}_2\cdots\mathrm{a}_N}=\sigma^{\mathrm{a}_1}_{\alpha_1}\sigma^{\mathrm{a}_2}_{\alpha_2}\cdots\sigma^{\mathrm{a}_N}_{\alpha_N}\delta^{\alpha_1\cdots\alpha_N}$$

Case $m=1$: Due to the symmetry of δ,

$$\frac{\partial g^{a_1a_2\cdots a_N}}{\partial x^d}=N\frac{\partial\sigma^{a_1}_{\alpha_1}}{\partial x^d}\sigma^{a_2}_{\alpha_2}\cdots\sigma^{a_N}_{\alpha_N}\delta^{\alpha_1\alpha_2\cdots\alpha_N}$$

Solving for $\partial\sigma/\partial x$, we get

$$\frac{\partial\sigma^{a_1}_{\alpha_1}}{\partial x^d}=\frac{1}{N}\tilde{\sigma}^{\alpha_2}_{a_2}\cdots\tilde{\sigma}^{\alpha_N}_{a_N}\delta_{\alpha_1\alpha_2\cdots\alpha_N}\frac{\partial g^{a_1a_2\cdots a_N}}{\partial x^d}$$

We multiply the equation by $\delta^{\alpha_1}_{\beta} = \sigma^b_\beta \tilde{\sigma}^{\alpha_1}_b$, and get

$$\frac{\partial \sigma^{a_1}_{\beta}}{\partial x^d} = \frac{1}{N} \sigma^b_\beta g_{ba_2a_3\cdots a_N} \frac{\partial g^{a_1a_2\cdots a_N}}{\partial x^d} \tag{1}$$

or

$$\partial^1 \sigma = \sigma f_1(\partial^{\{1\}} g)$$

To get a closer look at the relation between σ and its derivatives and g and its derivatives, we calculate now the relation for the following case:

Case $m = 2$:

$$\frac{\partial^2 \sigma^{a_1}_{\alpha_1}}{\partial x^{d_1} \partial x^{d_2}} = \frac{1}{N} \left(\frac{\partial \sigma^b_{\alpha_1}}{\partial x^{d_2}} g_{ba_2\cdots a_N} \frac{\partial g^{a_1a_2\cdots a_N}}{\partial x^{d_1}} \right.$$
$$+ \sigma^b_{\alpha_1} \frac{\partial g_{ba_2\cdots a_N}}{\partial x^{d_2}} \frac{\partial g^{a_1a_2\cdots a_N}}{\partial x^{d_1}}$$
$$\left. + \sigma^b_{\alpha_1} g_{ba_2\cdots a_N} \frac{\partial^2 g^{a_1a_2\cdots a_N}}{\partial x^{d_1} \partial x^{d_2}} \right)$$

Substituting the result (1) for $m = 1$, we get

$$\frac{\partial^2 \sigma^{a_1}_{\alpha_1}}{\partial x^{d_1} \partial x^{d_2}} = \frac{1}{N} \sigma^{b_1}_{\alpha_1} \left(\frac{1}{N} g_{b_1b_2\cdots b_N} g_{ba_2\cdots a_N} \frac{\partial g^{bb_2\cdots b_N}}{\partial x^{d_2}} \frac{\partial g^{a_1a_2\cdots a_N}}{\partial x^{d_1}} \right.$$
$$\left. + \frac{\partial g_{ba_2\cdots a_N}}{\partial x^{d_2}} \frac{\partial g^{a_1a_2\cdots a_N}}{\partial x^{d_1}} + g_{ba_2\cdots a_N} \frac{\partial^2 g^{a_1a_2\cdots a_N}}{\partial x^{d_1} \partial x^{d_2}} \right)$$

or

$$\partial^2 \sigma = \sigma f_2(\partial^{\{2\}} g)$$

Finally, we have the following case:

Case $m = n$: We assume

$$\partial^{n-1} \sigma = \sigma f_{n-1}(\partial^{\{n-1\}} g)$$

Then

$$\frac{\partial(\partial^{n-1}\sigma)}{\partial x^{d_n}} = \frac{\partial \sigma}{\partial x^{d_n}} f_{n-1} + \sigma \frac{\partial f_{n-1}}{\partial x^{d_n}}$$
$$= \frac{1}{N} \sigma g \frac{\partial g}{\partial x^{d_n}} f_{n-1} + \sigma \frac{\partial f_{n-1}}{\partial x^{d_n}}$$
$$= \sigma \left(\frac{1}{N} g \frac{\partial g}{\partial x^{d_n}} f_{n-1} + \frac{\partial f_{n-1}}{\partial x^{d_n}} \right)$$
$$= \sigma f_n \left(g, \ldots, \frac{\partial^n g}{\partial x^{d_1} \cdots \partial x^{d_n}} \right)$$

or

$$\partial^n \sigma = \sigma f_n(\partial^{\{n\}} g) \quad \blacksquare$$

Theorem 2.3.2. Any invariant analytic scalar function of σ and its first m derivatives is such a function of g and its first m derivatives alone.

Proof. Using the previous theorem, it suffices to prove that we can eliminate the σ dependence and leave only the dependence on g and its derivatives. It suffices to look at scalars made from polynomials in σ, $\tilde{\sigma}$, and $\partial_g^{\{m\}}$ by contraction. For such a polynomial to be a scalar invariant, each σ has to be paired with a $\tilde{\sigma}$. This results in a δ function. Therefore, only the dependence on $\partial^{\{m\}} g$ remains. $\blacksquare$

We have seen that all action densities that are invariant analytic scalar functions of the hyperspin vector field σ and its first m derivatives can be completely expressed in terms of the chronometric g and its first m derivatives. This simplifies some calculations. For the equations of motion, however, we must independently vary the components of σ, not g, which is subject to algebraic constraints (unless $N=2$). The hyperspin vector is hidden, but not gone.

2.4. Invariant Tensors

2.4.1. Notation

A sequence of indices $A_1 \cdots A_k$ is abbreviated by (A) and we define $|(A)| := k$. Most times k is understood from context and may vary. To reduce confusion, we mark by a prime any removed index. For example, the sequence (A) can be written as $A_p(A)'$ or as $A_{pq}(A)''$.

If there is a symmetry condition on the indices, it is indicated by the kind of brackets: a symmetric sequence of indices is $\{A\}$; an antisymmetric sequence is $[A]$, in agreement with the notation for anticommutators and commutators.

The following discussion deals only with proper spinors (undotted indices), but an analogous discussion can be carried out for the antispinor (with dotted indices).

At times indices of a tensor are omitted from the text to increase clarity and readability. To avoid confusion, we therefore never use the letter g for the determinant of the chronometric tensor, and whenever g appears it stands for the tensor itself.

2.4.2. Occurrence

If F is any index value, L is any tensor, and (A) and (B) are index value sequences, then by the *occurrence* of F in $L^{(A)}_{(B)}$ we mean the ordered pair $\binom{a}{b}$, where a is the number of F's in (A) and b is the number of F's in (B). By the *net occurrence* of F we mean $a-b$.

Example: The occurrence of 2 in L^{123}_{234} is $\binom{1}{1}$. The net occurrence is 0.

2.4.3. Representations

Let $D\binom{r}{s}$ be the representation of $SL(N, \mathbb{C})$ supported by the tensors of rank $\binom{r}{s}$.

Similarly, let $D^w\binom{r}{s}$ be the representations of $GL(N, \mathbb{C})$ supported by the relative tensor $L^{(A)}_{(B)}$ with $|(A)|=r$, $|(B)|=s$, and weight w. These transform under an element T of $GL(N, \mathbb{C})$ with determinant τ as follows:

$$L^{(A')}_{(B')}=\tau^w[(T^B_{B'})^{\otimes s}(\tilde{T}^{A'}_A)^{\otimes r}]\,L^{(A)}_{(B)} \qquad [\tilde{T}=(T^{-1})T] \tag{2}$$

where the superscript $\otimes r$ designates an r-fold tensor product. For short we write

$$L'=T\circ L$$

When restricted to $SL(N, \mathbb{C})$, $D^w\binom{r}{s}$ reduces to $D\binom{r}{s}$.

2.4.4. The Operator ∂

It is convenient to define the tensor operator ∂^F_H on a tensor $L^{(A)}_{(B)}$:

$$\begin{aligned}\partial^F_H(L^{(A)}_{(B)})&=\delta^{A_1}_H L^{FA_2\cdots A_r}_{(B)}+\delta^{A_2}_H L^{A_1F\cdots A_r}_{(B)}+\cdots+\delta^{A_r}_H L^{A_1A_2\cdots F}_{(B)}\\&\quad-(\delta^F_{B_1}L^{(A)}_{HB_2\cdots B_s}+\delta^F_{B_2}L^{(A)}_{B_1H\cdots B_s}+\cdots+\delta^F_{B_s}L^{(A)}_{B_1B_2\cdots H})\end{aligned} \tag{3}$$

Whenever $F=H$, ∂^F_H multiplies $L^{(A)}_{(B)}$ by the net occurrence $f-h$ of F in $L^{(A)}_{(B)}$:

$$F=H\Rightarrow\partial^F_H(L^{(A)}_{(B)})=(f-h)L^{(A)}_{(B)} \tag{4}$$

The notation ∂^F_H of (3) is to remind us that $\partial^H_F L$ is the Lie derivative of $T\circ L$ given by (2) with respect to T^F_H evaluated at $T^F_H=\delta^F_H$:

$$\partial^H_F L^{(A)}_{(B)}=\left.\frac{\partial(T\circ L)^{(A)}_{(B)}}{\partial T^F_H}\right|_{T^F_H=\delta^F_H}$$

2.4.5. Properties of Invariant Tensors

Lemma 2.4.1. Let L be an invariant $SL(N, \mathbb{C})$ tensor of rank $\binom{r}{s}$. The component $L^{(A)}_{(B)}$ is nonzero only if the net occurrence $a-b$ of any index value F in $L^{(A)}_{(B)}$ is the same for all F. If $L^{(A)}_{(B)}\neq 0$, then $r-s$ is an integer multiple of N.

Proof. A tensor is invariant under the unimodular group $SL(N, \mathbb{C})$ if

$$L^{(C)}_{(D)} = (\tilde{S}^{C}_{A})(S^{B}_{D}) L^{(A)}_{(B)}$$

for all $S \in SL(N, \mathbb{C})$, where $\tilde{S}^{A}_{C} S^{C}_{B} = \delta^{A}_{B}$ and $ABCD$ all refer to the same coordinate system.

It is convenient to extend any $SL(N, \mathbb{C})$ tensor L into a $GL(N, \mathbb{C})$ tensor L. For any $T \in GL(N, \mathbb{C})$, let us write $S^{A}_{B} = T^{A}_{B}/\tau^{(1/N)}$, where $\tau :=$ $\det(T^{A}_{B})$ and therefore $S \in SL(N, \mathbb{C})$. Then if $|(A)| = r$ and $|(B)| = s$, we define the action of T on L by using the action of S on L,

$$T \circ L = S \circ L \tag{5}$$

already defined in Section 2.4.1, with weight

$$w = (s-r)/N$$

Invariance of a function of L under $SL(N, \mathbb{C})$ now implies invariance under $GL(N, \mathbb{C})$. Therefore, we may differentiate (2) with respect to T^{F}_{H} and then choose $T^{F}_{H} = \delta^{F}_{H}$:

$$\frac{s-r}{N} \delta^{H}_{F} L^{(A)}_{(B)} + \partial^{H}_{F} L^{(A)}_{(B)} = 0 \tag{6}$$

Taking $H = F$ and applying (5), we get

$$\frac{r-s}{N} L^{(A)}_{(B)} = (f-h) L^{(A)}_{(B)}$$

Therefore $L^{(A)}_{(B)}$ is nonzero only if

$$(r-s)/N = f-h \tag{7}$$

The left-hand of (7) depends only on the dimension N of the space and on the rank $\binom{r}{s}$ of the given tensor L and note on F. Therefore $f-h$ must be the same for all index values F, and we set $f-h =: m$.

Equation (7) can now be rewritten as $r-s = mN$, where m is an integer. ■

The method of Lie derivative and the results (6), (7) are extensions to arbitrary N of Noriega and Schifini (1986). The first conclusion of Lemma 2.4.1, however, is stronger than theirs, even for $N = 2$.

Lemma 2.4.2. Any invariant tensors $L_{(A)}$ and $M^{(A)}$ with $|(A)| = N$ are scalar multiples of the antisymmetric tensors $\varepsilon^{[A]}$ and $\varepsilon_{[A]}$, respectively.

Proof. We use the case $m = 1$ of Lemma 2.4.1.

Choose any two index values A and B, we shall show that $L_{AB}(A) =$ $-L_{BA(A)}$.

Let $A = A' + B'$ and $B = A' - B'$; that is, we make a local coordinate transformation in which the new coordinate basis forms $e^{A'}$ are related to the old ones e^A by $e^A = e^{A'} + e^{B'}$ and $e^B = e^{A'} - e^{B'}$. We also remember that $L_A = L_{A'} e^{A'}_A + L_{B'} e^{B'}_A$, which we write as $L_{\langle A'+B'\rangle}$. We have

$$L_{AB(A)''} + L_{BA(A)''} = L_{\langle A'+B'\rangle\langle A'-B'\rangle(A)''} + L_{\langle A'-B'\rangle\langle A'+B'\rangle(A)''}$$
$$= 2(L_{A'A'(A)''} + L_{B'B'(A)''})$$

According to Lemma 2.4.1, $L_{A'A'(A)''} = L_{B'B'(A)''} = 0$ and therefore

$$L_{AB(A)''} = -L_{BA(A)''}$$

Thus, antisymmetry holds for any pair of indices.

The same proof can be carried out for the invariant tensor $M^{(A)}$. ■

Theorem 2.4.1. *An* $SL(N, \mathbb{C})$ tensor $L^{(A)}_{(B)}$ is invariant under $SL(N, \mathbb{C})$ iff L is a sum of tensor products of $\varepsilon^{[F]}$, $\varepsilon_{[G]}$, possibly with permuted or contracted indices.

Proof. "If": Trivial, since the ε's are invariant.

"Only if": We seek all tensors L that are invariant under $SL(N, \mathbb{C})$. We consider first a contravariant tensor L. From such L we form by (4) the $GL(N, \mathbb{C})$ relative tensor with weight $w = -m$, where m is defined in Lemma 2.4.1. If L is an $SL(N, \mathbb{C})$ tensor of rank $\binom{r}{0}$, then the corresponding $GL(N, \mathbb{C})$ tensor L transforms according to $D^{-m}\binom{r}{0}$.

If L is invariant under $SL(N, \mathbb{C})$, then L is invariant under $GL(N, \mathbb{C})$. If $L \neq 0$, then m is an integer by Lemma 2.4.1 and the subspace $\{\lambda L\}$ must support the one-dimensional representation $D^0\binom{r}{0}$. If we reduce the representation $D^{-m}\binom{r}{0}$ of $GL(N, \mathbb{C})$ into its irreducible parts (Young diagrams), each possible L corresponds to a one-dimensional part. It is easy to see that these are the Young diagrams consisting only of columns of length N, each column representing the tensor $\varepsilon^{[A]}$. This completes the proof for the contravariant case. (More detailed examination of the dimension of the representations of Young diagrams is given in Appendix A).

To prove the general case of a tensor L of rank $\binom{r}{s}$, we raise all the s covariant indices. Each index is raised by contraction of this index of the tensor L with one index of $\varepsilon^{[A]}$. By lemma 2.4.1, $r - s = mN$ if $L \neq 0$. Since index raising replaces each covariant index by $N-1$ contravariant indices, we can write

$$L\binom{r-s+Ns}{0} \overset{c}{=} L\binom{r}{s} \underbrace{\varepsilon^{[A]} \cdots \varepsilon^{[A]}}_{s \text{ factors}}$$

where C designates the above-described contraction and $(r - s + Ns)$ is also an integer multiple of N, as required for $L \neq 0$. To return from $L\binom{r-s+Ns}{0}$

to the original $L(^r_s)$, we contract each of the s antisymmetric groups of $N-1$ contravariant indices of $L(^{r-s+Ns}_0)$ with $\varepsilon^{[A]}$, and thus replace each group by one covariant index and then multiply by a normalization factor $1/N!$ to get back to $L(^r_s)$.

Thus, the ε's enable us to move freely between covariant and contravariant tensors, preserving the form asserted in the theorem. Therefore the above proof for the contravariant tensor suffices for all cases. ■

2.5. Hyperaction Lost

In the Kaluza–Klein theories the dimension of the manifold is fixed once and for all. In this program the dimension of the spin space and therefore of the spin manifold is a dynamical variable of the theory. We imagine that near singularities of the continuum theory higher dimensions may make a better approximate description of the universe (Finkelstein, 1986).

By a *hyperaction* I mean an expression in g and its derivatives that generates a family of actions, one for each N.

2.5.1. Geodesic and Normal Coordinates

The right coordinates can be quite crucial in the study of gravity and hypergravity. They can reduce the work immensely, simplify the formulas and calculations, and let the physics show through.

Most of the thinking processes of the rest of the work take place in *geodesic coordinates*, which are defined below. *Normal coordinates* also play an important role. At each point of time-space both coordinate systems can be defined.

Definition. Geodesic coordinate system (GSC) at x. A system of coordinates y such that the geodesics through x have linear parametric equations $y = v\tau$ with proper time τ as parameter. Then x is called the origin of the GCS.

Definition. GCS x^g is said to be *attached* to a coordinate system (CS) x^t if the coordinate lines of GCS are tangent to the corresponding coordinate lines of CS at the origin.

The choice of letter for vector and spinor indices is used to indicate the kind of coordinate system. A typical CS is x^t, while X^g is a CGS. The spinors underlying GCS carry the indices Γ and $\dot{\Gamma}$. The n equations relating the arbitrary CS to the GCS are

$$x^g = \dot{x}^t s$$

where s is a path parameter, $\dot{x}^t = (dp^t/ds)_0$, and $x^t = p^t(s)$ is the parametric equation of the geodesic with $p^t(0) = x_0$ and $p^t(1) = x^t$.

The geodesic equations

$$D^2P^t/Ds^2=0$$

where D is the covariant derivative with respect to the parameter s, become

$$d^2p^g/ds^2=0$$

in geodesic coordinates, where d is the ordinary derivative with respect to s.

There are also conditions on the ambispinor spaces. We define at each point x^t basis spinors ε^{Γ}, ε^{T}, $\varepsilon^{\dot{\Gamma}}$, and $\varepsilon^{\dot{T}}$. The relation between them is

$$\varepsilon^{\Gamma}(x)=X^{\Gamma}_{T}\varepsilon^{T}(x)$$

and similarily for the dotted indices. We impose at the origin

$$\varepsilon^{\Gamma}(x_0)=\varepsilon^{\Gamma}(x_0)\delta^{\Gamma}_{T}$$

and

$$D_s\varepsilon^{\Gamma}(p)=0$$

where D_s is the covariant derivative in the s direction.

Theorem 2.5.1. The law of transformation from deodesic coordinate system g to a coordinate system $\bar{g}$ is linear iff $\bar{g}$ is also geodesic.

Proof. Clear. ■

Definition. Following Thomas (1934), we define *the mth extension* of $\sigma^{g}_{\dot{\Gamma}\Gamma}$ as a tensor whose component in a GCS are

$$\sigma^{g}_{\dot{\Gamma}\Gamma|(m)}=\partial^m\sigma^{g}_{\dot{\Gamma}\Gamma|0}=:\partial_{\{a\}}\sigma^{g}_{\dot{\Gamma}\Gamma}$$

When transforming to any CS x^t the mth extension becomes

$$\sigma^{t}_{\dot{T}T|(t)}=\sigma^{g}_{\dot{\Gamma}\Gamma|(m)}X^{\dot{\Gamma}}_{T}X^{\Gamma}_{T}X^{g}_{t}X^{(m)}_{(t)}$$

Definition. Normal coordinates are understood differently by different writers. We follow Veblen and von Neumann (1935) and define normal coordinates as a coordinate system in which the σ mapping takes a specific standard form. For $N=2$ it is customary to take as normal coordinates those that use the four basic Pauli matrices for the four components of σ. For higher dimension a different choice proves to be more convenient.

2.5.2. Second-Order Equations Only for $n=4$

This is now the time to ask if we can find an action density for Bergmann manifolds that leads to equations of motion linear in the second derivative of the chronommetric tensor g. We shall see that the answer is no.

Lemma 2.5.1. Let $L(\partial^{\{m\}}\sigma)$ be an invariant Lagrangian of any order. Then the Euler-Lagrange derivative

$$E := \frac{dL}{d\sigma} := \frac{\partial L}{\partial \sigma} - \partial_x \frac{\partial L}{\partial(\partial_x \sigma)} + \partial_x^2 \frac{\partial L}{\partial(\partial_x^2 \sigma)} \pm \cdots$$

is a tensor.

Proof:

$$\delta L = \int_v (dL/d\sigma)\, \delta\sigma\, d^n x$$
$$+ \int_{\partial v} (\text{expression resulting from integration by parts})\, \delta\sigma \qquad (8)$$

We choose a surface on which $\delta\sigma = 0$ and (8) becomes

$$\delta L = \int_v (dL/d\sigma)\, \delta\sigma\, d^n x \qquad (9)$$

If we have the transformation $\sigma \to T \circ \sigma$ under which σ transforms as

$$\sigma' = X\Lambda\sigma\Lambda^H \qquad (x' = Xx;\ \psi' = \Lambda\psi)$$

then

$$\delta\sigma' = X\Lambda\delta\sigma\Lambda^H$$

and

$$\delta L = \int_v (dL/d\sigma')\, \delta\sigma'\, d^n x' = \int_v (dL/d\sigma') X\Lambda\delta\sigma\Lambda^H X^n\, d^n x \qquad (10)$$

By comparing (9) and (10) we get the tensorial character of $E = dL/d\sigma$. ■

Lemma 2.5.2. The coefficient of $\partial^2\sigma$ in the quasilinear invariant Lagrangian $L(\partial^{\{2\}}\sigma)$ is an invariant tensor under $GL(N, \mathbb{C})$ transformations from one geodesic normal coordinate system to another.

Proof. We choose a geodesic coordinate system ($\partial\sigma_{10} = 0$) and we take it to be normal at the origin 0: we may write (Veblen and von Neumann (1935)

$$\sigma^{m}_{A\dot{A}10} := \sigma^{M\dot{M}}_{A\dot{A}10} = \delta^{M}_{A} \otimes \delta^{\dot{M}}_{\dot{A}}$$

In such coordinate systems

$$L = K^{ln\dot{A}A}_{m} (\partial^2\sigma)^{m}_{A\dot{A}nl}$$

where K is a tensor, since L is a tensor for all values of $\partial^2\sigma$, and $\partial^2\sigma$ is a tensor.

The invariance of K stems immediately from the definition of invariance:

$$L(\sigma) = L(T \circ \sigma) = K(T \circ \partial^2\sigma) \tag{11}$$

On the other hand,

$$L = K\partial^2\sigma = (T \circ K)(T \circ \partial^2\sigma) \tag{12}$$

Comparing (11) and (12) gives us the desired result

$$K = T \circ K \quad \blacksquare$$

Theorem 2.5.2. An invariant action density that is linear in the second derivative of σ exists only for four-dimensional Bergmann manifolds.

Proof. Let $L = L(\sigma, \partial\sigma, \partial^2\sigma)$ be such an action density. Following Veblen and von Neumann (1935), we can always transform to a geodesic coordinate system in which $\partial\sigma = 0$ and $L = L(\sigma, \partial^2\sigma)$. The requirement that L is linear in $\partial^2\sigma$ means that the coefficient K of $\partial^2\sigma$ (which is an invariant tensor by Lemma 2.5.2) does not depend on $\partial^2\sigma$. Thus

$$K = \partial L/\partial(\partial^2\sigma)$$

and since the index structure of $\partial^2\sigma$ is $\binom{1}{3}\times\binom{1}{3}$, the index structure of the tensor K is $\binom{3}{1}\times\binom{3}{1}$. By Lemma 2.4.1, K vanishes unless $N = 2$ and $n = N^2 = 4$. $\blacksquare$

2.5.3. Generalized Zumino Action Only for $n = 4$

Giving up the quest for a hyperaction density that leads to equations of motion linear in the second derivative of g, we search for a hyperaction that reduces to the Einstein–Hilbert action for $N = 2$, but leads to equations of motion in degree higher than two. Therefore, while Zumino writes his candidate action as an algebraic expression in the curvature tensor (which is the commutator of the covariant derivatives), we use the covariant derivative D^a_b explicitly. D^a_b is a 1-form differential operator (with the coordinate index hidden)

$$D^a_b = \delta^a_b\partial + \omega^a_b$$

where ω^a_b is the connection. We still use the exterior product of Zumino.

To construct a hyperaction density candidate from an exterior product of D's, we need to take into account the special features of hyperspin manifolds. As was pointed out before, the Bergmann chronometric tensor carries N indices and when lowering an index with it we end up with $N - 1$ lower indices. To have an operator with all its indices lowered, we need to contract a product of D^a_b's with a product of $g_{a_1 \ldots a_N}$'s. Any such product has a nonzero contribution when contracted with the Levi-Civita tensor ε only if it has not more than one index inherited from g (since g is symmetric). Therefore, to make an operator with all its indices lowered, we must lower $N-1$ upper indices with one g to give us one lower index. Thus, we contract $N-1$ of the D_b^a's with one g and antisymmetrize to construct an $(N-1)$-form $D_{ba_1 \cdots a_{N-1}}$:

$$D_{ba_1 \cdots a_{N-1}} = \delta_{a_1 a_2 \cdots a_{N-1}}^{a_{1'} a_{2'} \cdots a_{N-1'}} g_{b_1 \cdots b_{N-1} b} D_{a_{1'}}^{b_1} \wedge D_{a_{2'}}^{b_2} \wedge \cdots \wedge D_{aN-1'}^{bN-1} \tag{13}$$

We next study the operator (13) more closely and discover that it is an algebraic expression rather than a proper differential operator only for an even number of D's.

Lemma 2.5.3. The operator (13) is algebraic for a Bergmann manifold with an odd dimension N and is a nonalgebraic differential operator for a manifold with an even dimension N.

Proof. If we write the expression (13) explicitly, we get an unsymmetrized sum of products of $M = N-1$ terms, each looking like $(\delta_j^i \partial + \omega_j^i)$. The commutator $[\partial, \omega] = (\partial \cdot \omega)$, the derivative of ω, is algebraic. Therefore, if in a product of several D's we shift the derivatives to the right, we make only an algebraic contribution. We can therefore assume that after some manipulations we are left with expressions that have all their derivatives concentrated at the right. Then each term that has more than one derivative vanishes in the process of the antisymmetrization. The only terms that deserve a closer look are those that have only one derivative in them.

As an exercise, we first look at two cases.

$N=2$, $M=1$: It is clear that the nonalgebraic contribution is $\delta_j^i \partial$, a proper differential operator.

$N=4$, $M=3$: Schematically, we look at antisymmetrization of the sum

$$\partial_1 \omega_2 \omega_3 + \omega_1 \partial_2 \omega_3 + \omega_1 \omega_2 \partial_3$$

which is

$$\begin{aligned}
&\partial_1 \omega_2 \omega_3 + \partial_3 \omega_1 \omega_2 + \partial_2 \omega_3 \omega_1 - \partial_2 \omega_1 \omega_3 - \partial_3 \omega_2 \omega_1 - \partial_1 \omega_3 \omega_2 \\
&\quad + \omega_1 \partial_2 \omega_3 + \omega_3 \partial_1 \omega_2 + \omega_2 \partial_3 \omega_1 - \omega_2 \partial_1 \omega_3 - \omega_3 \partial_2 \omega_1 - \omega_1 \partial_3 \omega_2 \\
&\quad + \omega_1 \omega_2 \partial_3 + + \omega_3 \omega_1 \partial_2 + \omega_2 \omega_3 \partial_1 + \omega_2 \omega_1 \partial_3 - \omega_3 \omega_2 \partial_1 - \omega_1 \omega_3 \partial_2
\end{aligned}$$

Each term on the first line can be paired with a term on the second line to make a commutator, which is algebraic, i.e.,

$$\partial_1 \omega_2 \omega_3 - \omega_2 \partial_1 \omega_3 = (\partial_1 \omega_2 - \omega_2 \partial_1) \omega_3 = (\partial_1 \cdot \omega_2) \omega_3$$

The terms on the third line, however, have no such term to pair with, and since the operators ω_i and ω_j in general do not commute, the third line makes a proper differential-operator contribution.

Arbitrary N: As in the example of $N=4$, $M=3$, it is obvious that for an even M we get pairs that are commutators and therefore make only algebraic contributions, while for odd M we are left with unpaired differential terms. ■

Theorem 2.5.3. A hyperaction that is a product of operators of the kind defined in (13) and that reduces to a Lovelock-Zumino action for $N=2$ does not exist for Bergmann manifolds of $N>2$.

Proof. We use the operator (13) as a building block for a density action. We try

$$D_{a_1\cdots a_N} \wedge D_{b_1\cdots b_N} \wedge \cdots \wedge e_k \wedge e_1 \wedge \cdots \varepsilon^{a_1\cdots a_N b_1\cdots b_N \cdots k_1 \cdots} \tag{14}$$

with s of the D's and r of the e's. Three requirements are to be satisfied simultaneously:

(i) Due to Lemma 2.5.3, if the dimension N is even, s should be even. Otherwise, there is no restriction on s. The proof that for even N and even S (14) is algebraic is similar to the proof of Lemma 2.5.3.

(ii) $Ns+r=n$(for n −ad indices).

(iii) $(N-1)s+(N-1)r=n=N^2$ (for coordinate indices).

The condition (iii) for $N\neq 1$ becomes

$$s+r=N^2/(N-1)$$

Since $s+r$ has to be a positive integer and since N^2 and $N-1$ differ in parity, the above equation has a solution only for $N=2$. This is the case where (13) is Zumino's action for $N=2$.

In other words, the generalized Zumino hyperaction exists only for $n=4$ and not for higher dimensions for a Bergmann manifold. ■

3. HYPERGRAVITY

3.1. Hyperaction Regained

Simply attempting to generalize Riemannian actions to Bergmann manifolds leads to a dead end. Either there are too many possibilities or none.

The scalar wave equation is of order N in Bergmann manifolds, since to make a scalar from the differential operator $\partial_{\dot{A}A}$ we need to take its norm. This leads us to seek gravitational actions that are quasilinear of order N instead of 2.

The hyperaction density on Bergmann manifolds B_N that we first discovered is proportional to the Nth extension of the chronometric tensor. We show below that

$$L_N = \mathrm{p} g^{\{a\}}|_{\{a\}} \tag{15}$$

where ρ is the chronometric density already defined, reduces to the Einstein–Hilbert action for $N=2$. The notation $\{a\}$ for a collective index is discussed in Section 2.4.1.

Since a GCS is geodesic at but one point, we cannot easily integrate this expression over the manifold to get the action from the action density. We therefore look for other candidates. We learn from (13) to expect differential order N.

A more tractable action density of this kind is

$$L_N = \rho R_{a_1 a_2; a_3 \cdots a_N} g^{\{a\}} \tag{16}$$

where $R_{a_1 a_2}$ is the Ricci tensor, and the semicolon designates covariant derivative. Partial derivative is designated by a comma. Sometimes we apply the collective index notation for the $N-2$ derivatives of the Ricci tensor and write it as $R_{(a)}$.

We conjecture that two quasilinear Nth-order actions differ at most by a constant multiplier and terms of lower order.

Theorem 3.1. The two hyperactions (15) and (16) reduce (up to a numerical coefficient) to the curvature scalar density R for $N=2$.

Proof. (i) *The action (15)*: For $N=2$

$$R_{(a)} = R_{a_1 a_2} g^{a_1 a_2} = R \quad \blacksquare$$

(ii) *The action (16)*: For $N=2$ we show that $R = 3g^{\{a\}}|_{\{a\}} = 3g^{ab}|_{ab}$. Some identities are needed for the proof:

Thomas (1934) proves that in GCS

$$g_{ab,cd} + g_{ad,bc} + g_{ac,db} = 0 \tag{7}$$

where $g_{ab,cd} = \partial_c \partial_d g_{ab}$.

Using the identity (17) and differentiating twice the relations

$$g^{ab} g_{ac} = \delta^b_c$$

we get the identity

$$g_{ab,cd} = -g^{nm}_{\ \ ,cd} g_{na} g_{mb}$$

from which we can also deduce

$$g_{ac,bd} g^{ab} g^{cd} = g_{ad,bc} g^{ab} g^{cd}$$

and

$$g^{ab}_{\ \ ,ab} = -g^{ab}_{\ \ ab,}$$

Now, in GCS

$$\begin{aligned} R^a_{bcd} &= \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} \\ &= g^{am}(\partial_c \Gamma_{mbd} - \partial_d \Gamma_{mbc}) \\ &= \tfrac{1}{2} g^{am}(\partial_c [g_{mb,d} + g_{md,b} - g_{bd,m}] - \partial_d [g_{mb,c} + g_{mc,b} - g_{bc,m}]) \\ &= \tfrac{1}{2} g^{am}\{g_{md,bc} - g_{bd,mc} - g_{cm,bd} + g_{ab,md}\} \end{aligned}$$

Since

$$R = R_{bd} g^{bd} = R^a_{bad} g^{bd}$$

we have

$$\begin{aligned} R &= \tfrac{1}{2} g^{am} g^{bd}\{g_{md,ba} - g_{bd,ma} - g_{am,bd} + g_{ab,md}\} \\ &= -g^{am} g^{bd}\{g_{ab,md} - g_{bd,ma}\} \end{aligned} \tag{18}$$

If we multiply and contract (17) with $-g^{am} g^{bd}$, we get

$$g^{am} g^{bd}\{g_{bd,ma} + g_{ba,md} + g_{bm,ad}\} = g^{am} g^{bd}\{g_{bd,ma} + 2g_{ab,md}\} = 0$$

Solving (19) for $g^{am} g^{bd} g_{bd,ma}$ and substituting in (18), we get

$$R = -3g^{am} g^{bd} g_{ab,md} = 3g^{md},\ md$$

Renaming indices and transforming from GCS to CS, we can write

$$R = -3g^{ab}|_{ab} \quad \blacksquare$$

3.2. Equations of Motion in Hyperspin Manifolds

In deriving the equations of motion, it is necessary to recall Section 2.3, in which we prove that while the action can be expressed in terms of the chronometric tensor g and its derivatives alone, in performing variations, σ has to be taken as the fundamental variable.

The action we choose is the one of (16):

$$\delta \int \rho L_N (dx)^n = \int \rho(\delta R_{(a)}) g^{\{a\}}(dx)^n + \int \rho R_{(a)} \delta g^{\{a\}}(dx)^n + \int (\delta\rho) R_{(a)} g^{\{a\}}(dx)^n \tag{20}$$

In a GCS at x, $\Gamma = 0$, $\delta r \Gamma_{;a} = \delta r \Gamma_{,a}$, and

$$\delta R_{a_1 a_2} = \delta \Gamma^b_{a_1 a_2; b} - \delta \Gamma^b_{a_1 b; a_2}$$

at x. Since this is a tensor equation, it holds in any CS, and since x is arbitrary, it holds everywhere. Therefore we may differentiate it:

$$\delta R_{a_1 a_2; \{a\}''} = \delta \Gamma^b_{a_1 a_2; b\{a\}''} - \delta \Gamma^b_{a_1 b; a_2\{a\}''} \tag{21}$$

To calculate the first integral of (20), we apply (21) and then integrate by parts $N-2$ times. The result is zero, since Γ vanishes in GCS and the boundary terms vanish as well. Therefore, as in the case $N=2$, the variations of R_{ab} do not contribute to the equation of motion. It suffices to vary only the simpler factor ρg^{ab} of the action. This fact, well known for $N=2$, underlies the Zumino analysis, which therefore can likely be carried out for any N, with suitable changes.

The other two integrals are to be rewritten in terms of $\delta\sigma$. For this we need to calculate $\delta\rho/\delta\sigma$ and $\delta g/\delta\sigma$:

$$\begin{aligned}\rho &= \det[(\sigma^{b}_{\dot{B}B})^{-1}]\\ \delta\rho &= -\delta(\varepsilon_{[b]}\sigma^{b2}_{\dot{1}1}\sigma^{b2}_{\dot{1}2}\cdots\sigma^{b_n}_{\dot{N}N})\end{aligned}\tag{22}$$

When we apply the variation δ in (22), the first term in the sum of n terms is

$$-\varepsilon_{[b]}\sigma^{b_2}_{\dot{1}2}\cdots\sigma^{b_n}_{\dot{N}N}\delta\sigma^{b_1}_{\dot{1}1} = -(\text{cofactor of }\delta\sigma^{b_1}_{\dot{1}1})\delta\sigma^{b_1}_{\dot{1}1}$$

and the sum of all the terms gives

$$\delta\rho = -\det[(\sigma^{b}_{\dot{B}B})^{-1}]\sigma^{A\dot{A}}_{a}\delta\sigma^{a}_{\dot{A}A} = -\rho\sigma^{A\dot{A}}_{a}\delta\sigma^{a}_{\dot{A}A}\tag{23}$$

$$g^{\{a\}} = \sigma^{a_1}_{\dot{A}_1A_1}\cdots\sigma^{a_N}_{\dot{A}_NA_N}\varepsilon^{[A]}\varepsilon^{[\dot{A}]} = \{\sigma^{a}_{\dot{A}A}\}\varepsilon^{[A]}\varepsilon^{[\dot{A}]}$$

and due to the symmetry of g

$$\delta g^{\{a\}} = N\delta\sigma^{a}_{\dot{A}A}\{\sigma^{a}_{\dot{A}A}\}'\varepsilon^{[A]}\varepsilon^{[\dot{A}]}\tag{24}$$

We can now substitute (23) and (22) in (20) and equate the coefficients of $\delta\sigma^{a}_{\dot{A}A}$ to zero. This leads to the equation

$$N\rho R_{(a)}\{\sigma^{a}_{\dot{A}A}\}'\varepsilon^{[A]}\varepsilon^{[\dot{A}]} - \rho\sigma^{A\dot{A}}_{a}R = 0\tag{25}$$

where we define

$$R := R_{(a)}g^{\{a\}}$$

We now divide (25) by $N\rho$ and multiply and contract with $\sigma^{b}_{\dot{A}A}$ to get the equation of motion in vacuum

$$R^{b}_{a} - \frac{1}{N}\delta^{b}_{a}R = 0\tag{26}$$

where we define

$$R^{b}_{a} := R_{(a)}g^{\{a\}'b}$$

The group manifold of the unitary group $U(N)$ may be provided with a natural invariant hyperspin structure and is then homogeneous of constant curvature (Holm, 1987). Therefore, for $N>2$, R and R^{b}_{a} (which include covariant derivatives of the curvature) vanish, and equation (26) is satisfied. For $N=2$, as we know from Einstein, R and R_a^b are nonzero and dust and the cosmological term must be added in order to have $U(2)$ obey the equation of motion.

3.3. Hypergravity with Sources

When we study gravity in the presence of sources, the action (16) has to be modified. We assume a field φ and take the action density L' to be a function of φ and g and their derivatives. Therefore equation (20) is replaced by

$$\delta \int \rho L_N (dx)^n + \delta \int \rho L'(dx)^n = 0$$

where $L' = L'(\varphi, g)$.

We define the *energy* (momentum stress) *tensor* T_a^b:

$$T_a^b := -\frac{1}{N\rho} \sigma^b_{\dot{A}A} \frac{\delta(\rho L')}{\delta \sigma^a_{\dot{A}A}}$$

In a way similar to the derivation of the equations of motion in vacuum, we get the equations of motion in the presence of matter

$$R_a^b - \frac{1}{N} \delta_a^b R = T_a^b$$

Considering the difference in order, the resemblance to Einstein's field equation is noteworthy. It is also remarkable that even though the fundamental variable of the theory is the spin-vector σ, there is no evidence of it in the equations of motion. The "soldering" is hidden and what we see is only the chronometric tensor.

3.4. Gauge Particles on Bergmann Manifolds

What kind of quanta are present in the gravitational field of a Bergmann manifold? This is a quantum mechanical question. To deal with it, we consider a flat manifold with small perturbation $\delta\sigma$. The equations of motion for $\delta\sigma$ are linearized versions of the general law of motion for σ already given in Section 3.2. In a normal GCS we may assign indices $E=1,2$ to external spinor components and $I=3,\ldots,N$ to internal ones. (We choose the word *external* for four time-space dimensions and *internal* for the higher dimensions). Then external tangent vector components have the index structure $v^{E\dot{E}}$, the internal tangent vector is $v^{I\dot{I}}$ and the mixed vector is $v^{E\dot{I}}$ or $v^{I\dot{E}}$.

We use the representations of the Lorentz group (Paerl, 1969) to classify the particles that arise from the field $\delta\sigma$.

The representations of the Lorentz group are $D(J, \dot{J})$, where

$$2J = \text{the number of spinor indices}$$

$$2\dot{J} = \text{the number of antispinor indices}$$

Not all indices contribute to the (ordinary four-dimensional) "external" spin. Contributions to the external spin come only from the external dimensions ($N = 1, 2$; $\dot{N} = \dot{1}, \dot{2}$). The internal dimensions are to account for other degrees of freedom, such as electric charge, color, flavor, etc. Therefore J and $\dot{J}$ range from 0 (no external indices) to 1 (two external indices) and $J\dot{J}$ is 00, $0\frac{1}{2}$, $\frac{1}{2}0$, 01, 10, $\frac{1}{2}\frac{1}{2}$, $1\frac{1}{2}$, $\frac{1}{2}1$, or 11. This means that the theory predicts at most gauge particles of spin, 0, 1/2, 1, 3/2, and 2. This corrects the erroneous prediction of higher spin by Finkelstein (1986) made on the basis of the transformation property of $g^{\{a\}}$, not σ.

We get a closer look at the index partition in the following table:

	$E\dot{E}$	$E\dot{I}$	$I\dot{E}$	$I\dot{I}$
$E\dot{E}$	$D(1, 1)$	$D(1, \frac{1}{2})$	$D(\frac{1}{2}, 1)$	$D(\frac{1}{2}, \frac{1}{2})$
$E\dot{I}$	$D(1, \frac{1}{2})$	$D(1, 0)$	$D(\frac{1}{2}, \frac{1}{2})$	$D(\frac{1}{2}, I)$
$I\dot{E}$	$D(\frac{1}{2}, 1)$	$D(\frac{1}{2}, \frac{1}{2})$	$D(0, 1)$	$D(0, \frac{1}{2})$
$I\dot{I}$	$D(\frac{1}{2}, \frac{1}{2})$	$D(\frac{1}{2}, 0)$	$D(0, \frac{1}{2})$	$D(0, 0)$

where the rows stand for the sesquispinor indices of $\delta\sigma$ and the columns for the time-space indices of $\delta\sigma$.

On its face, the theory has room for gravitons $D(1, 1)$, electromagnetic field ($D(1, 0) \dotplus D(0, 1) \dotplus D(\frac{1}{2}, \frac{1}{2})$), spin-$\frac{1}{2}$ particles ($D(\frac{1}{2}, 0) \dotplus D(0, \frac{1}{2})$), spin-$\frac{3}{2}$ particles (($D(\frac{1}{2}, 1) \dotplus D(1, \frac{1}{2}) \dotplus D(\frac{1}{2}, 0) \dotplus D(0, \frac{1}{2})$), and scalar fields $D(0, 0)$. We should now follow the clue given by the transformation properties of the fields and study the physical properties of their quanta in order to verify the above identification.

3.5. Future Problems

This work opens many questions about hypergravity. Here I point out only four:

(i) Section 2.1 presented two hyperactions $g^{\{a\}}|_{\{a\}}$ and R. Do all hyperactions quasilinear in the Nth order differ at most by a constant multiplier and terms of lower degrees?

(ii) Do hypergravitational black and white holes exist? If so, what is their geometry and topology?

(iii) How do we account for our experience of only four time-space dimensions?

(iv) What is the linearized quantum theory? Although the variable σ likely has only statistical meaning, the quantized theory might have meaning at low energies.

4. CONCLUSIONS

We have investigated the relations between the hyperspin vector σ and the chronometric tensor g in Bergmann manifolds of all possible dimensions. Though σ is the fundamental entity, it may without loss of information be replaced by g in any invariant action density. However, σ still plays an indispensable role as the fundamental variable to be varied in the process of deriving the equations of motion from an action.

The only invariant tensors in a Bergmann manifold are the obvious ones, linear combinations of products of Kronecker and Levi-Civita tensors and a scalar density.

Unlike the Riemannian case, where quasilinear differential equations of the second differential order exist for any dimension, in Bergmann manifolds such an equation can be constructed only for $n=4$.

Zumino's actions generalize from Riemannian to Bergmann manifolds only for $n=4$.

We propose two actions for Bergmann manifolds; for $N=2$ they are proportional to the Einstein–Hilbert action. From one of the proposed actions (R), we derive the equations of motion in vacuum; then, assuming a nongravitational field φ on the manifold, we define the energy tensor and derive the equations of motion for a Bergmann manifold with sources.

A preliminary analysis based on a small perturbation $\delta\sigma$ to a flat background describes gauge quanta with spins 0, 1/2, 1, 3/2, and 2 for any N.

It now seems possible and physically interesting to do for arbitrary N all the spinorial physics that has been done in recent decades for $N=2$.

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